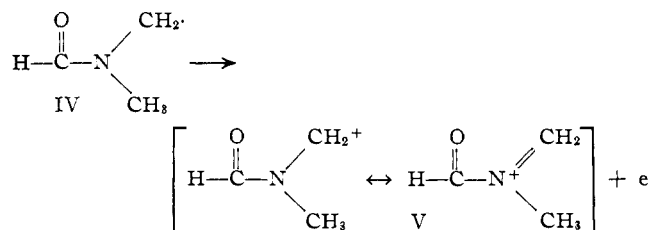


the ether II resulting from hydrolysis of I which may be shown to be very facile. Alternatively, the radical IV might discharge another electron at the anode to give the carbonium ion V, which could then react with solvent to give I and III or with water to give N-hydroxymethyl-N-methylformamide, the probable precursor of II.



SPRAGUE RESEARCH CENTER  
SPRAGUE ELECTRIC COMPANY  
NORTH ADAMS, MASSACHUSETTS

SIDNEY D. ROSS  
MANUEL FINKELSTEIN  
RAYMOND C. PETERSEN

RECEIVED MARCH 17, 1964

### On the Numerical Integration of Rate Equations

Sir:

A recent publication<sup>1</sup> by Schaad describes a Monte Carlo method, of considerable generality, for the integration of kinetic rate equations whose analytical solutions are intractable or nonexistent. It is the purpose of this communication to point out that "direct" numerical methods<sup>2,3</sup> are usually to be preferred because of their much greater speed, greater accuracy, and equal ease of programming. In particular, the fourth-order Runge-Kutta method<sup>3,4</sup> is very convenient for kinetic problems. This method, while roughly half as fast as more efficient methods, has for kinetic work the advantages of being self-starting, not requiring integration at equally spaced values of time, and being readily available as a library routine at most computer installations.

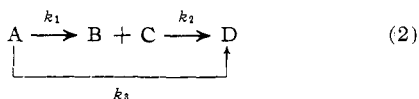
A general chemical reaction scheme may be specified by a set of equations of the form

$$\frac{dC_i}{dt} = F_i(t, T, C_0, \dots, C_n) \quad (1)$$

$$i = 0, 1, \dots, n$$

where  $C_i$  is the concentration of the  $i$ th reacting species, and the  $F_i$ , functions of time, temperature, and concentrations of all species. It is precisely systems of this type which may be integrated directly, for specific initial conditions, by the Runge-Kutta method. Temperature will usually enter  $F_i$  only as a parameter, but if it is a nonconstant function of time (or indeed of any of the concentrations), the method is still applicable; hence nonisothermal problems, to which the Monte Carlo method is unsuited, may be treated.

As an example, the system treated by Schaad<sup>1</sup>



(1) L. J. Schaad, *J. Am. Chem. Soc.*, **85**, 3588 (1963).

(2) L. Fox, "Numerical Integration of Ordinary and Partial Differential Equations," Pergamon Press, New York, N.Y., 1962.

(3) A. Ralston and H. S. Wilf, "Mathematical Methods for Digital Computers," John Wiley and Sons, Inc., 1960, Chapters 8 and 9.

(4) S. Gill, *Proc. Cambridge Phil. Soc.*, **47**, 96 (1951).

which gives, for  $B(0) = C(0)$  the system

$$\frac{dA}{dt} = -(k_1 + k_3)A$$

$$\frac{dB}{dt} = k_1A - k_2B^2 \quad (3)$$

$$\frac{dD}{dt} = k_3A + k_2B^2$$

was integrated with  $k_1 = 1$ ,  $k_2 = 0.5$ ,  $k_3 = 0.2$  from the initial condition  $A(0) = 2$ ,  $B(0) = C(0) = D(0) = 0$  (all units arbitrary). The integration was carried out from  $t = 0$  to  $t = 5$  with a step of 0.1, results agreeing with the exact solution given by Pearson, King, and Langer<sup>5</sup> to five figures throughout. This required approximately 2.5 sec. on a Ferranti Mercury computer,  $1/500$  of the time required for the Monte Carlo method on a computer of similar speed.<sup>6</sup>

Using the Runge-Kutta method, it is possible and practicable to determine rate constants by adjusting trial values until the solution of the differential equations agrees as well as possible with experimental measurements. For a least-squares determination, one has to minimize the quantity

$$D = \sum_i \sum_j w_{ij} (\Delta_{ij})^2 \quad (4)$$

where  $\Delta_{ij}$  is the difference between the calculated and experimental concentrations of the  $i$ th species at the  $j$ th experimental point. The  $w_{ij}$  are weights assigned to compensate for differences in magnitude of the various concentrations and differences of precision among the various measurements. In particular, if the  $k$ th species is not measured in the experiment,  $w_{kj} = 0$  for all  $j$ .

Two procedures which have been found suitable for the minimization of  $D$  with respect to the rate constants are those devised by Rosenbrock<sup>7</sup> and Fletcher and Powell.<sup>8</sup> The latter has more rapid convergence but requires the evaluation of the partial derivatives of  $D$  with respect to the various rate constants. These in turn require calculation of the partial derivatives of the calculated concentrations with respect to the rate constants at all experimental points, giving rise to further differential equations which must be integrated simultaneously with the rate equations.

To illustrate these processes, both were used to extract the original  $k$ 's from the values, obtained above, of  $B$  as a function of  $t$ . These values were rounded to three figures and regarded as "experimental data." Two starting approximations to the  $k$  values were used; the first,  $k_1 = 0.95$ ,  $k_2 = 0.47$ ,  $k_3 = 0.25$ , was obtained graphically from the  $B$  vs.  $t$  data; the second was taken five times poorer, *viz.*,  $k_1 = 0.75$ ,  $k_2 = 0.35$ ,  $k_3 = 0.45$ . All weights were taken equal to unity.

The Rosenbrock method gave the final result  $k_1 = 1.0000$ ,  $k_2 = 0.5001$ ,  $k_3 = 0.1997$ , standard deviation =  $2.6 \times 10^{-4}$ , from both starting approximations. From the better approximation, this required 143 integrations of the rate equations, and from the poorer

(5) R. G. Pearson, L. C. King, and S. H. Langer, *J. Am. Chem. Soc.*, **73**, 4149 (1951).

(6) The Monte Carlo method required 11 min. on an IBM-7072.

(7) H. H. Rosenbrock, *Computer J.*, **3**, 175 (1960).

(8) R. Fletcher and M. J. D. Powell, *ibid.*, **6**, 163 (1963).

240, 6.5 and 10 min. Mercury time being used for the two determinations.

The Fletcher-Powell minimization required, in this case, integration of a set of seven differential equations. For these, each integration over 50 points required 5.5 sec., but the same results as given by Rosenbrock's method were obtained with 22 and 31 integrations, consuming 2.1 and 2.7 min., respectively.

It may be seen from these examples that either minimization procedure, coupled with Runge-Kutta integration, yields a suitable method of determining rate constants when the rate equations cannot be integrated. In general, the Fletcher-Powell procedure will, because of its greater speed, be preferred for routine use after a mechanism has been established. The Rosenbrock method is usually preferable, however, for the preliminary investigation of a problem, as much less alteration in the computer program is required when a different set of rate equations is to be tried. Alternatively, preliminary investigation might be done on an analog computer, as the qualitative effect of variations in the rate constants could be easily determined. For determining the best values of the rate constants, however, digital computation is clearly superior, as the bookkeeping can be done by the machine. On an analog computer, the variation of the rate constants would have to be carried out by the operator, a procedure of very low efficiency, particularly if more than two constants are to be varied.

DEPARTMENT OF CHEMISTRY  
IMPERIAL COLLEGE  
LONDON, S.W.7, ENGLAND

IAN D. GAY

RECEIVED MARCH 23, 1964

### The Association of Tetrabutylammonium Bromide in Methanol-Nitrobenzene Mixtures

Sir:

In a recent publication,<sup>1</sup> a molecular interpretation was presented to explain the abnormal association behavior of tetrabutylammonium bromide ( $\text{Bu}_4\text{NBr}$ ) in methanol-nitrobenzene mixtures. It was claimed that association constants for this electrolyte in a large number of solvent systems gave a single straight line when the log was plotted against the reciprocal dielectric constant. The point for pure methanol was on this line, but as nitrobenzene was added, the association constant showed a minimum and finally increased to a point above the line for pure nitrobenzene. This behavior could not be explained by coulombic interactions since both solvents have approximately the same dielectric constant. The enhanced  $K_A$  for  $\text{Bu}_4\text{NBr}$  in the nitrobenzene-rich mixtures was explained by a specific interaction between a nitrobenzene molecule and a  $\text{Bu}_4\text{NBr}$  ion pair. In the methanol-rich mixtures a specific solvation of  $\text{Br}^-$  ion by methanol was postulated.

We believe that the minimum described above is an artifact due in part to the data and in part to the method used to calculate  $K_A$ . The association constants considered were those of Sadek and Fuoss<sup>2</sup> and were obtained from their conductance data by means of the Fuoss-Shedlovsky method. These data

cover far too narrow a concentration range ( $1-8 \times 10^{-4} N$ ) for the accurate determination of association constants. Furthermore, in the Fuoss-Shedlovsky method, the Onsager limiting conductance equation is used to evaluate the conductance of the free ions. This procedure is known to be completely unreliable for  $K_A$  much less than 100. The association constants under consideration here ranged from about 26 for methanol to 45 for nitrobenzene solutions. The more recent conductance data for this salt in pure methanol<sup>3</sup> were not considered nor was the much more recent analysis<sup>4</sup> based on the Fuoss-Onsager conductance theory.<sup>5</sup> In that analysis, Fuoss claimed definite association in both pure solvent components but no association for the intermediate mixtures. However, the not too reliable  $\Delta_A$  method was used in that analysis. An ion size  $\bar{a} = 6$  was found to fit the data in other solvent mixtures of lower dielectric constant where association was substantial. Fuoss used this same value of  $\bar{a}$  to eliminate one of the unknowns in the conductance equation. This assumption of a constant  $\bar{a}$  for every solvent mixture is extremely risky since it has been shown to be invalid for dioxane-water mixtures.<sup>6</sup>

We have analyzed the data<sup>2,3</sup> by the Fuoss-Onsager conductance theory<sup>7</sup> on an IBM 7070 computer using a Fortran program similar to that described by Kay.<sup>8</sup> The early data<sup>2</sup> for  $\text{Bu}_4\text{NBr}$  in methanol solution gave an extremely low  $\bar{a}$  and poor precision when treated as an unassociated electrolyte and a negative value of  $\bar{a}$  when treated as an associated electrolyte. The 1954 data for methanol solutions, on the other hand, gave  $\Lambda_0 = 96.20 \pm 0.03$  and  $\bar{a} = 2.8 \pm 0.3$  with a standard deviation of 0.03 when  $\text{Bu}_4\text{NBr}$  was treated as unassociated and gave a negative  $K_A$  when treated as an associated electrolyte. These latter results indicate that  $\text{Bu}_4\text{NBr}$  is slightly, if at all, associated ( $K_A < 10$ ) in methanol.

In order to verify this result, precise conductance measurements were carried out for  $\text{Bu}_4\text{NBr}$  in anhydrous methanol over the concentration range  $3-47 \times 10^{-4} M$ . The actual conductances will be reported with a more comprehensive study of the quaternary ammonium salts in  $\text{H}_2\text{O}$ ,  $\text{D}_2\text{O}$ , and methanol at a later date. The results of a Fuoss-Onsager analysis<sup>7</sup> of two runs are recorded here in Table I where the first two entries resulted from treating  $\text{Bu}_4\text{NBr}$  as an unassociated electrolyte while the last two entries resulted from treating  $\text{Bu}_4\text{NBr}$  as an associated electrolyte. These results indicate, as found above with the older data, that  $\text{Bu}_4\text{NBr}$  is in fact associated only to a very slight extent, if at all, in methanol. The association constant of about 3.5 could be the result of about a 10% error in the evaluation of the electrophoretic effect, or a small amount of  $c^{1/2}$  dependence that has been neglected in the conductance theory. In any case  $K_A$  is considerably less than 26, the value used by Hyne, and there is no doubt that association of this salt in methanol solutions is considerably less than that predicted from the association behavior in solvent mixtures of much lower dielectric constant.

(3) H. Sadek and R. M. Fuoss, *ibid.*, **76**, 5897 (1954).

(4) H. Sadek and R. M. Fuoss, *ibid.*, **81**, 4507 (1959).

(5) L. Onsager and R. M. Fuoss, *J. Phys. Chem.*, **61**, 668 (1957).

(6) J. E. Lind, Jr., and R. M. Fuoss, *ibid.*, **65**, 999, 1414 (1961).

(7) R. M. Fuoss and F. Accascina, "Electrolyte Conductance," Interscience Publishers, Inc., New York, N. Y., 1959.

(8) R. I. Kay, *J. Am. Chem. Soc.*, **82**, 2099 (1960).

(1) J. B. Hyne, *J. Am. Chem. Soc.*, **85**, 304 (1963).

(2) H. Sadek and R. M. Fuoss, *ibid.*, **72**, 301 (1950).